Math 206B Lecture 6 Notes

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1 Examples, Dimension, and Irreducibility of S^λ

1.1 Examples of S^λ

Let t be a tabloid, and consider $\mathbb{C}\langle \sigma \cdot e_t : \sigma \in S_n \rangle = S^\lambda$.

Example 1.1. Let $\lambda = (n)$. There is a single poly-tabloid,

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & \cdots & n \\ \hline \end{array}$$

S_n acts trivially on this, so $S^{(n)} = 1$.

Example 1.2. Let $\lambda = (1^n)$. The poly-tabloids correspond to permutations:

$$\sigma \mapsto \begin{array}{|c|}\hline \sigma(1) \\ \hline \sigma(2) \\ \hline \vdots \\ \hline \sigma(n) \\ \hline \end{array}$$

Let $\sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\sigma = \mathcal{X} \in \mathbb{C}[S_n]$. Then $\pi \cdot \mathcal{X} = \operatorname{sign}(\pi)\mathcal{X}$.

Example 1.3. Let $\nu_k = (n-k, 1^k)$. Then $M^{\nu_k} = \operatorname{ind}_{S_{n-k} \times 1 \times \cdots \times 1}^{S_n} 1$. The dimension is $\dim(M^{\nu_k}) = n!/(n-k)!$. What is S^{ν_k}? Permutations do not act on the columns, but they permute the elements in the first column. Let $v = \sum_{\sigma \in S_{k+1}} \operatorname{sign}(\sigma)\sigma$. What is $\pi \cdot v = w$? In general $\mathbb{C}\langle \pi e_t \rangle$ will be a representation of S_n.

Example 1.4. Let $\lambda = (2,2)$. Let

$$t = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$$

Then

$$v = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array} + \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 2 & 4 \\ \hline\end{array} - \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 4 & 2 \\ \hline\end{array} + \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline\end{array}$$

Then $(1\,2)v = v$ and $(3\,4)v = v$. We can also calculate

$$(3\,4)v = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array} + \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & 4 \\ \hline\end{array} - \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 3 \\ \hline\end{array} + \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline\end{array} := w.$$

Then M^μ is a 2-dimensional vector space spanned by v, w.

1.2 Dimension and irreducibility of S^λ

Lemma 1.1. *Let S^λ be as above.*

1. $\dim(S^\lambda) = f^\lambda = \#\,\mathrm{SYT}(\lambda)$.
2. $n! \leq \sum_\lambda (f^\lambda)^2$, *where equality holds iff S^λ is irreducible for each λ.*

Definition 1.1. The **dominance order** on partitions is the partial order $\lambda \trianglelefteq \mu$ if $\lambda_1 + \cdots + \lambda_k \geq \mu_1 + \cdots + \mu_k$ for all k.

We had that $M^\lambda = \bigoplus_{\mu \trianglelefteq \lambda} m_{\lambda,\mu} S^\mu$. The proof of the inequality in the 2nd statement comes from this fact.

How should we prove the equality in statement 2 of the lemma? Suppose G is a finite group. We have both a left and a right action of $G \circlearrowright \mathbb{C}[G]$. This gives us a $G \times G$ representation, $\bigoplus_\pi \pi \otimes \pi$, where the sum is over all irreducible representations. We want to view this equation as $n!$ being the order of the group, and the $(f^\lambda)^2$ terms being the tensor product of two irreducible representations. It is hard to do this in general, but it will work in the symmetric group. This is called the RSK correspondence.